Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol. 28, N. 1, 73-87

Flatness of Gaussian curvature and area of ideal triangles

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Abstract. Let M be a C^k, $k \geq 4$, compact surface of genus greater than two whose curvature is negative in all points but along a simple closed geodesic $\gamma(t)$ where the curvature is zero at every point. We show that the area of ideal triangles having a lifting of γ as an edge is infinite. This provides a family of surfaces having ideal triangles of infinite area whose geodesic flows are equivalent to Anosov flows, in contrast with the well-known examples of surfaces with flat strips which also have ideal triangles of infinite area. By the CAT-comparison theory we can deduce, using these surfaces as models, that a C^∞ compact surface of non-positive curvature having one geodesic along which the curvature is zero has ideal triangles of infinite area.

Introduction

In the present work we investigate the area of ideal triangles of non-hyperbolic metrics in the disk having non-positive curvature. We consider a compact surface M of non-positive curvature and genus greater than two, so the universal covering $\tilde{M}$ is the disk endowed with the pull-back of the metric. It is well known that $\tilde{M}$ admits a compactification $\tilde{M}(\infty)$ where $\partial\tilde{M}(\infty)$ corresponds to the collection of asymptotic directions of geodesics in $\tilde{M}$. Given $a, b, c \in \partial\tilde{M}(\infty)$ an *ideal triangle* with vertices a, b, c is the region of $\tilde{M}$ bounded by the geodesics joining a with b, b to c and c to a. A very nice result by Barges and Ghys [**3**] tells us that if the area of ideal triangles is finite and constant then the curvature of M is constant and negative. In [**5**] this result is shown to be false for C^3 surfaces if we simply ask the existence of an upper bound for the area of ideal triangles. There are counterexamples constructed

Received 3 October 1996.
Partially supported by CNPq of Brazilian Government

by gluing annulus of revolution, having negative curvature at all points but along a simple closed geodesic, and a piece of surface of constant negative curvature. In fact, given any $\alpha > 0$ this construction provides a $C^{2+\epsilon}$ Riemannian metric of non-positive curvature, where $\epsilon = \frac{2}{\alpha}$ having a non-hyperbolic geodesic $\gamma(t)$ such that every asymptotic geodesic $\beta(t)$ of $\gamma(t)$ satisfies

$$d(\gamma(t), \beta(t)) \leq C \frac{1}{t^{\alpha}}$$

In particular, when $\alpha = \frac{3}{2}$ we get a C^3 surface with finite integral behavior of the function $f(t) = d(\gamma(t), \beta(t))$. In this construction appears an explicit connection between α, the finiteness (or not) of the area of ideal triangles and the way the curvature goes to zero near $\gamma(t)$. More precisely, the curvature $K(p)$ at points p close to $\gamma(t)$ along geodesics normal to $\gamma(t)$ satisfies the following formula:

$$K(p) = -y(p)^{\epsilon} f(y(p))$$

where $y(p)$ is the distance to the geodesic $\gamma(t)$, $f(y)$ is an analytic function with $f(0) \neq 0$. So for instance, if $\alpha = \frac{3}{2}$, then $\epsilon = \frac{4}{3}$ and in general, since α must be greater than 1 in order to obtain finite area of ideal triangles we have $\epsilon < 2$. This is why the counterexamples are but C^3. The main result of this paper is to show that for C^k metrics, $k \geq 4$, the conjecture holds:

Theorem 1. *Let M be a C^3 compact surface having negative curvature at all points but along a simple closed geodesic $\gamma(t)$, where the curvature is zero for every t. Let N be a normal neighborhood of γ and let $y : N \longrightarrow R$ be the distance y(p) from every point of N to γ. Suppose that the Gaussian curvature K restricted to N satisfies*

$$K(p) = -y(p)^{\epsilon} f(y(p))$$

where $\epsilon \geq 2$ and $f(y)$ is an analytic function with $f(0) = 1$. Then the area of ideal triangles having γ as an edge is infinite.

Theorem 1 tells us that if the curvature near $\gamma(t)$ is suitably flat then the area of ideal triangles is not finite. Notice that, if the surface were of class C^k with $k \geq 4$, the number ϵ in Theorem 1 has to be at least

2, so Theorem 1 holds for C^k surfaces where $k \geq 4$. We present here a short, general argument connecting the flatness of the curvature and the convergence of the area of the saturation by the geodesic flow of pieces of the stable manifold of $\gamma(t)$. It uses some estimates of the solutions of the Ricatti equation along non-hyperbolic geodesics that measure the rate of decay of stable Jacobi fields near to a zero curvature geodesic. We would like to point out that Theorem 1 implies that there is a family of surfaces having ideal triangles of infinite area in the universal covering whose geodesics flows are equivalent to Anosov flows, in contrast with the classical examples of metrics having flat strips which clearly posses ideal triangles of infinite area. Besides, these surfaces have ergodic geodesic flows and Pesin's sets of total measure (i.e., the set of orbits in the unit tangent bundle having non-zero Lyapunov exponents) so their ergodic metric properties are very similar to those of Anosov geodesic flows.

Two final remarks before ending the introduction. The existence of the above mentioned C^3 counterexamples (thus having C^2 geodesic flows) indicates that something more than soft arguments is needed to show Theorem 1. On the other hand, by the triangle comparison theory of Cartan, Alexandrov, Toponogov we can deduce that every C^∞ compact surface of non-positive curvature having one geodesic along which the curvature is zero has ideal triangles of infinite area. We shall prove this fact in the last section (section 4) using as a comparison model an annulus of nonpositive curvature satisfying the hypothesis of Theorem 1 and having minimal exponent $\epsilon = 2$. So we can conclude that

Theorem 2. *In the category of C^∞ compact surfaces of non-positive curvature, the geodesic flow is Anosov if and only if every ideal triangle in the universal covering of the surface has finite area.*

Indeed, since by Theorem 1 the existence of a zero curvature geodesic implies the existence of ideal triangles of infinite area, we can easily show that the finiteness of the area of every ideal triangle grants the existence of $T > 0$, $\epsilon > 0$ and $\delta > 0$ such that the following holds:

Given any geodesic $\eta(t)$ parameterized by arclength each subsegment of $\eta(t)$ of the form $\eta[a, a+T]$ passes through a region of negative curvature $K < -\delta$ during an interval of time of length at least ϵ.

Now, it is not hard to see that under these conditions the geodesic flow is Anosov.

1. Preliminaries of the geometry of non-positive curvature

In this section we shall state some notations and elementary asymptotic properties of geodesics in non-positive curvature manifolds. All geodesics will be parameterized by arc length. $\tilde{M}$ will denote the universal covering of M endowed with the lifting of the metric of M.

Definition 1.. Let M be a Riemannian manifold of non-positive curvature. Two geodesics $\gamma(t)$, $\beta(t)$ in $\tilde{\mathrm{M}}$ are called asymptotic if there exists $\mathrm{C} > 0$ such that

$$d(\gamma(t), \beta(t)) \;\leq\; C$$

for every $t \geq 0$.

The geodesics γ and β are called *bi-asymptotic* if there exists $C > 0$ such that

$$d(\gamma(t), \beta(t)) \;\leq\; C$$

for every $t \in R$.

Due to the convexity of the metric, given a geodesic γ in $\tilde{M}$ and $p \in \tilde{M}$ there is a geodesic asymptotic to γ containing p. Two different bi-asymptotic geodesics bound a flat strip in $\tilde{M}$, so we deduce that in the hypotheses of the main theorem the universal covering $\tilde{M}$ of our surface M does not contain bi-asymptotic geodesics. Moreover, we have that the distance between two asymptotic geodesics decreases to 0 as $t \to +\infty$. In fact, the distance between asymptotic geodesics decreases uniformly on compact sets in the sense of [**5**].

The *horosphere* $H(\gamma(t))$ of a given geodesic $\gamma \subset \tilde{M}$ at the point $\gamma(t)$ is constructed as follows:

Let $S_{r-t}(\gamma(r))$ for $r > t$ be the sphere of radius $r - t$ centered at $\gamma(r)$. It contains the point $\gamma(t)$ for every r and letting $r \to +\infty$ we obtain a limit submanifold, the horosphere $H(\gamma(t))$, characterized by the fact that a geodesic is asymptotic to γ if and only if it is orthogonal somewhere to a horosphere of γ (and then to every horosphere).

We shall always parametrize an asymptote $\beta(t)$ of $\gamma(t)$ in a way that $\beta(t) = \beta \cap H(\gamma(t)$.

Definition 2. Given $\theta = (p, v) \in T_1\tilde{M}$-the unit tangent bundle of $\tilde{M}$-the *Buseman function* $b_\theta : \tilde{M} \longrightarrow R$ associated to θ is defined by

$$b_\theta(x) = \lim_{t \to +\infty} (d(x, \gamma(t)) - t)$$

where $\gamma(t)$ the geodesic whose initial conditions are $\gamma(0) = p, \gamma'(0) = v$.

Busemann functions are C^2 [**7**], its gradient ∇b_θ is always tangent to the asymptotes of $\gamma(t)$ and its level hypersurfaces are precisely the horospheres of γ. We denote by $\psi_t : \tilde{M} \longrightarrow \tilde{M}$ the flow of this gradient vector field. Then we have that

$$\psi_t(H(\gamma(s))) = H(\gamma(s+t))$$

for every s, t. For more details and a good exposition of the basic theory of manifolds of non-positive curvature see [**2**].

2. The Ricatti equation

The purpose of this section is to investigate the solutions of the well-known Ricatti equation associated to the Jacobi equation near the zero-curvature geodesic γ. Recall that given a geodesic $\beta(t)$, the Jacobi equation

$$J''(t) + K(t)J(t) = 0$$

has two special kinds of solutions without zeros:

- The "stable" solutions $J_v^s(t)$ where $v \in T_{\beta(0)}M$ is orthogonal to $\beta(t)$ and $J_v^s(0) = v$. These solutions are characterized by the fact that

  $$| J_v^s(t) | \leq | v |$$

  for every $t \geq 0$.
- The "unstable" solutions $J_v^u(t)$ where v is as before, $J_v^u(0) = v$ and

  $$| J_v^u(t) | \leq | v |$$

  for every $t \leq 0$.

The functions

$$u^s(t) = \frac{J_v^{s\prime}(t)}{J_v^s(t)},$$
$$u^u(t) = \frac{J_v^{u\prime}(t)}{J_v^u(t)}$$

satisfy the differential equation

$$u'(t) + u^2(t) + K(t) = 0$$

(it is not hard to see that u^s and u^u do not depend on the vector v). These functions are the geodesic curvatures of the stable (resp. unstable) horospheres of $\beta(t)$ and notice that

$$\frac{d}{dt}\log(J_v^s(t)) = u^s(t)$$

so their means provide the Lyapunov exponents of the geodesic flow of M. We are interested in the behavior of these functions near a non-hyperbolic geodesic of M. To simplify the notation, let us denote $u^s(t) = u(t)$. First we need to obtain a sharper version of the usual comparison theorems (see [**6**] for instance) for the Ricatti equation applicable to our case:

Lemma 2.1. *Let M be a surface of non-positive curvature. Let $\beta(t)$ be a geodesic of M and suppose that*

$$lim_{t\to+\infty}\frac{1}{t}\int_0^t K(x)dx = 0$$

Then the following assertion holds:

If there exists t_0 such that $u^2(t_0) > -K(t_0)$ then there exists $t_1 > t_0$ with $u^2(t_1) < -K(t_1)$.

Proof. First, notice that the set of points where $u^2(t) > -K(t)$ is just the set of points where $u'(t) = -u^2(t) - K(t)$ is negative. Therefore, if there exists t_0 such that $u^2(t_0) > -K(t_0)$ there exists an interval (a, b) where the same holds, and we can assume without loss of generality that (a, b) is a maximal connected component of R with the property that $\forall t \in (a, b)$, $u^2(t) \geq -K(t)$. Then we have that $u^2(a) = -K(a)$, $u^2(b) = -K(b)$ and thus a, b are a critical points of $u(t)$. Moreover, from the choice of a, b we deduce that a is a local maximum of $u(t)$ and b is a local minimum of $u(t)$. Now, by contradiction, assume that b is infinite. Then $u(t)$ would be a strictly decreasing function for every $t \geq a$, and since $u(t) \leq 0$ for every $t \in R$ (this is because the curvature is non-positive) we would get some $c > 0$ such that

$$\lim_{t \to +\infty} u(t) = -c$$

so we would have

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t u(s) ds \leq -c$$

and therefore

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t K(s) ds = lim_{t \to +\infty} \frac{1}{t} \int_0^t -u^2(s) ds \leq -c^2$$

contradicting the hypotheses on the decay of the curvature. □

Corollary 2.1. *In the hypotheses of Lemma 2.1, if there exists $x \in R$ such that $K(t)$ is also non-decreasing and non-zero for every $t \geq x$ then*

$$| u(t) | \leq (-K(t))^{\frac{1}{2}}$$

for every $t \geq x$.

Proof. Suppose, by contradiction, that $-u(t) > (-K(t))^{\frac{1}{2}}$ for some $t \geq x$. Then $u^2(t) > -K(t)$ and from Lemma 2.1 we get that there exist $a < t < b$ such that $u^2(a) = -K(a)$, $u^2(b) = -K(b)$ and $u'(t) \leq 0$ $\forall t \in [a, b]$. Moreover, a would be a local maximum of $u(t)$ and b a local minimum of $u(t)$, and since $u(t)$ decreases in $[a, b]$ we would have that

$$-K(a) = u^2(a) < u^2(b) = -K(b)$$

(recall that $u(t)$ is non-positive) so $K(a) > K(b)$ contradicting the assumption on the curvature. □

3. The proof of the main theorem

So let M be a surface satisfying the hypotheses of Theorem 1 and let $\gamma(t)$ be the zero-curvature closed geodesic. Let N be the normal, tubular neighborhood of $\gamma(t)$ satisfying

$$K(p) = -y(p)^\alpha f(y(p))$$

$\forall p \in N$, where $y(p)$ is the distance from p to γ. We begin by stating a technical lemma about quasi-geodesic properties of curves having bounded curvature.

Lemma 3.1. *Let $f : (-a, a) \longrightarrow M$ be a C^2 curve parameterized by arc length such that its geodesic curvature $k(t)$ satisfies $\mid k(t) \mid \leq k_0$. Then there exist $\delta_0 > 0$ and $k_1 > 0$ depending on k_0 such that if $l(f(t,s))$ is the length of $f(t,s)$ we have*

$$\mid d(f(t), f(s)) - l(f(t,s)) \mid \leq k_1 d(f(t), f(s))^2$$

for every $\mid t - s \mid \leq \delta_0$. Moreover, we have that $k_1 \to 0$ if $k_0 \to 0$.

For the proof it suffices to look at graphs of functions $f : R \longrightarrow R$. It follows after some elementary calculus with the local form of f and we shall not write it down here. In few words, a curve with bounded second derivatives resembles locally a geodesic, in the sense that we can compare (up to a factor) its length with the distance between pair of points in the curve.

Lemma 3.2. *Let $\beta(t)$ be asymptotic to $\gamma(t)$. Then there exists $\epsilon > 0$ and $T > 0$ such that*

- *The curvature $K(\beta(t))$ is strictly increasing for $t \geq T$.*
- $lim_{t \to +\infty} \frac{1}{t} \int_0^t K(\beta(s))ds = 0$.
- *The following estimates hold for every $t \geq T$:*

$$(1 - \epsilon)d(\gamma(t), \beta(t))^\alpha \leq -K(\beta(t)) \leq (1 + \epsilon)d(\gamma(t), \beta(t))^\alpha.$$

Proof. We shall only sketch the proof of this fact because the arguments are quite standard. To simplify things let us argue in $\tilde{M}$. So let us identify $\gamma(t)$ with some lifting in $\tilde{M}$. Notice that, from the remarks of section 1 the distance between $\beta(t)$ and $\gamma(t)$ decreases to 0 as $t \to +\infty$. So there exists $T > 0$ such that $\beta(t) \in N$ $\forall t \geq T$. From the hypotheses in K we deduce that there exists a tubular neighborhood $N' \subset N$ of γ such that the curvature in N' is negative and decreasing with respect to the distance to $\gamma(t)$. Assume for simplicity that $N' = N$. Then the composition

$$K(\beta(t)) = -(d(\beta(t), \gamma))^\alpha f(d(\beta(t), \gamma))$$

is an increasing function of t for $t \geq T$. This proves item 1. Item 2 is a straightforward consequence of the fact that $\beta(t)$ is asymptotic to $\gamma(t)$ where the curvature is zero for every t. By shrinking the width of N if necessary, we have that given $\epsilon > 0$ small enough the following holds:

$$(1-\epsilon)d(\beta(t),\gamma(t)) \leq d(\beta(t),\gamma) \leq d(\beta(t),\gamma(t)).$$

Indeed, this is a consequence of lemma 3.1. The points $\gamma(t)$ and $\beta(t)$ belong to the same horosphere $H(\gamma(t))$ for every t. So if $d(\beta(t),\gamma(t))$ is small enough the segment of $H(\gamma(t))$ between them approaches the geodesic segment lying between $\gamma(t)$ and $\beta(t)$ since the geodesic curvature of $H(\gamma(t))$ is close to zero in N (recall that $H(\gamma(t))$ is perpendicular to $\gamma(t)$ at every $t \in R$). This is due to the fact that the geodesic curvature of horospheres is given by the solutions of the Ricatti equation, and these solutions tend to zero as the distance to $\gamma(t)$ goes to zero according to Corollary 2.1. So we finally obtain that the distance $d(\beta(t),\gamma)$ is "efficiently" approached by $d(\beta(t),\gamma(t))$ for $t \geq T$.

Now, item 3 follows from the above remark and the hypotheses on the curvature K in the region N. $\square$

Let $c : [0,a] \longrightarrow N$ be the subcurve of the connected component C of $H(\gamma(0)) \cap N$ which is perpendicular to γ at $\gamma(0)$ and lies between $c(0) = \gamma(0)$ and one of the boundary points $c(a)$ of C. Assume that $c[0,a]$ is the arc length parameterization. For a given lifting $\bar{\gamma}$ of $\gamma(t)$ in $\tilde{M}$ let $\psi_t : \tilde{M} \longrightarrow \tilde{M}$ be the Busemann flow of $\bar{\gamma}(t)$. This flow induces a flow $\bar{\psi}_t$ in N everywhere tangent to the asymptotes of γ, and it satisfies $\bar{\psi}_t(N) \subset N$ for $t \geq 0$. To simplify the notation we identify ψ and $\bar{\psi}$. Let $P : C \longrightarrow C$ be the Poincaré map of this flow restricted to C (notice that $P(C) \subset C$ and $P(c[0,a]) \subset c[0,a]$ because this flow is contracting). Let L be the period of $\gamma(t)$ (notice that $P = \psi_L$) and let $\gamma_s(t)$ be the asymptote of $\gamma(t)$ defined by $\gamma_s(0) = c(s)$ for every $s \in [0,a]$. Then we have the following estimate for the length of $P(c[0,a])$:

Lemma 3.3. *Let $l(P(c[0,a]))$ be the length of $P(c[0,a])$. Then*

$$l(P(c[0,a])) \geq l(c[0,a])e^{-L(1+\epsilon)^{\frac{1}{2}}d(\gamma_a(0),\gamma(0))^{\frac{\alpha}{2}}}.$$

Proof. We can estimate the length of $P(c[0,a])$ by means of the following formula:

$$l(P(c[0,a])) = \int_0^a D\psi_L(c'(s))ds$$

where $D\psi_t$ is the differential of the map ψ_t. From the construction of stable Jacobi fields we deduce that

$$D\psi_t(c'(x)) = J^s_{c'(x)}(t)$$

so we get

$$\begin{aligned} l(P(c[0,a])) &= \int_0^a D\psi_L(c'(s))ds \\ &= \int_0^a J^s_{c'(x)}(L)dx \\ &= \int_0^a c'(x)e^{\int_0^L u_x(y)dy}dx \end{aligned}$$

where

$$u_x(y) = \frac{J^{s\prime}_{c'(x)}(y)}{J^s_{c'(x)}(y)}.$$

Notice that in the last identity we used that $J^s_{c'(x)}(0) = c'(x)$. Since $\gamma_x(t) \in N$ for every $t \geq 0$ we can apply Lemma 3.1 and Corollary 2.1 to deduce that

$$\begin{aligned} \mid u_x(t) \mid &\leq (-K(\gamma_x(t)))^{\frac{1}{2}} \\ &\leq (1+\epsilon)^{\frac{1}{2}} d(\gamma_x(t),\gamma(t))^{\frac{\alpha}{2}} \end{aligned}$$

$\forall t \geq 0$, $x \in [0,a]$ and, since $u_x(t) \leq 0$ for every x and t we have

$$u_x(t) \geq -(1+\epsilon)^{\frac{1}{2}} d(\gamma_x(t),\gamma(t))^{\frac{\alpha}{2}}.$$

So we obtain

$$\begin{aligned} l(P(c[0,a])) &\geq \int_0^a c'(x)e^{\int_0^L -(1+\epsilon)^{\frac{1}{2}} d(\gamma_x(y),\gamma(y))^{\frac{\alpha}{2}} dy}dx \\ &\geq \int_0^a c'(x)e^{-L(1+\epsilon)^{\frac{1}{2}} sup_{\{y\in[0,L],x\in[0,a]\}} d(\gamma_x(y),\gamma(y))^{\frac{\alpha}{2}}} dx \\ &\geq l(c[0,a])e^{-L(1+\epsilon)^{\frac{1}{2}} d(\gamma_a(0),\gamma(0))^{\frac{\alpha}{2}}} \end{aligned}$$

as we wished to prove. □

Proof of Theorem 1:
Let $P^n(c[0,a]) = c[0,a_n]$ be the n^{th} iterate of $c[0,a]$ by the Poincaré map P. From Lemma 3.1 we have that there exists $\delta > 0$ small, depending on N such that

$$d(\gamma_x(t), \gamma(t)) \leq l(\psi_t(c[a,0])) \leq (1+\delta) d(\gamma_x(t), \gamma(t))$$

for every $x \in [0,a], t \geq 0$. Let $B = L(1+\epsilon)^{\frac{1}{2}}(1+\delta)^{\frac{\alpha}{2}}$. From Lemma 3.2 we obtain

$$l(c[0,a_n]) \geq l(c[0,a_{n-1}])e^{-B(l(c[0,a_{n-1}]))^{\frac{\alpha}{2}}}.$$

We can assume without loss of generality that $l(c[0,a]) < 1$ (and therefore $l(c[0,a_n]) < 1 \forall n$, and since $\alpha \geq 2$ by hypotheses we get

$$\begin{aligned} l(c[0,a_n]) &\geq l(c[0,a_{n-1}])e^{-Bl(c[0,a_{n-1}])} \\ &\geq l(c[0,a_{n-1}])(1 - Bl(c[0,a_{n-1}])) \end{aligned}$$

where in the last inequality we used that $e^x \geq 1+x$. Therefore, we have the following formula:

$$l(c[0,a_n]) \geq l(c[0,a]) \prod_{i=1}^{n-1}(1 - Bl(c[0,a_i])).$$

So suppose by contradiction that the area of the region $S = \cup_{t \geq 0}\psi_t(c[0,a])$ is finite (here, again, we identify $c[0,a]$ with one of its lifting by the covering map). This would imply that

$$\sum_{i=0}^{\infty} l(c[0,a_i]) < \infty.$$

On one hand, by the last inequality we would have that

$$\sum_{i=0}^{\infty} l(c[0,a_i]) \geq l(c[0,a]) \sum_{k=0}^{\infty}\prod_{i=1}^{k}(1 - Bl(c[0,a_i]))$$

so $\prod_{i=1}^{k}(1 - Bl(c[0,a_i])) \longrightarrow 0$ as $k \to +\infty$ and this infinite product would "diverge to 0" (as said in the argot of the theory of infinite products). But on the other hand, an infinite product of the type $\prod_{i=0}^{\infty}(1+b_i)$

is convergent and non-vanishing if and only if the series $\sum_{i=0}^{\infty} b_i$ is convergent, which clearly leads to a contradiction. This implies Theorem 1 since the liftings of the region S in $\tilde{M}$ are included in the ideal triangles having the liftings of $\gamma(t)$ as one of their edges.

4. Comparison theory and the proof of Theorem 2

The purpose of the section is to show the following result:

Lemma 4.1. *Let (M, g) be a C^∞ compact surface of nonpositive curvature, and suppose that there exists a geodesic $\gamma(t)$ of M along which $K(\gamma(t)) = 0$ $\forall t \in R$. Then every ideal triangle having any lifting of $\gamma(t)$ as an edge has infinite area.*

We shall just sketch the main steps of the proof of Lemma 4.1 since it follows from Theorem 1 and standard arguments in comparison theory.

Step 1: Model surface.
Given $c > 0$ it is possible to construct an annulus (M_c, g_c) of nonpositive curvature with a zero curvature closed geodesic $\gamma(t)$ such that the curvature in a tubular neighborhood of γ has the formula

$$K_c(p) = -y(p)^2 f_c(y(p))$$

where $f_c : R \longrightarrow R$ is a positive analytic function satisfying $f_c(0) = c$, and $y(p)$ is the distance from p to γ. It is not hard to get such annulus using the theory of surfaces of revolution, but we are going to use an example described in [**1**] that is very simple for the calculations. Let $A = S^1 \times R$ endowed with the metric $g(s,t) = Y^2(t)ds^2 + dt^2$, where $Y(t) = 1 + at^4$. It is easy to check that the lines $\gamma_s : R \longrightarrow A$ given by $\gamma_s(t) = (s,t)$ are geodesics with arclength parameter t, the circle $\gamma^0(s) = (s, 0)$ is also a geodesic of A, and the curvature of A is given by

$$K(s,t) = \frac{-Y''(t)}{Y(t)} = \frac{-12at^2}{1 + at^4}$$

Since $\mid t \mid = \mid t(p) \mid$ is the distance from $p = (s,t) \in A$ to the geodesic γ^0, and $K(s,t) = 0$ if and only if $t = 0$, the geodesic γ^0 has a tubular neighborhood N satisfying the hypothesis of Theorem 1. Now, taking $a = \frac{c}{12}$ we obtain the metric g_c.

Step 2: The comparison with the model surface.
Let (M, g) be a C^∞ compact surface of nonpositive curvature having a zero curvature geodesic $\gamma(s)$. Let $\beta(s)$ be any lifting of $\gamma(s)$ in $\tilde{M}$ and consider a normal tubular neighborhood $N_\epsilon(\beta)$ in $\tilde{M}$. Let us take Fermi coordinates in $N_\epsilon(\beta)$, $F : R \times (-\epsilon, \epsilon) \longrightarrow N_\epsilon(\beta)$, $F(s,t) = \exp_{\beta(s)}(tn(s))$, where $n(s)$ is a differentiable unit vector field everywhere normal to $\beta'(s)$. Let $A_g = (R \times (-\epsilon, \epsilon), F^*g)$ be the strip $R \times (-\epsilon, \epsilon)$ endowed with the pullback by F of the metric g. By the Taylor's formula and the compactness of M, there exists $\epsilon_1 \leq \epsilon$, $C > 0$, such that for every $s \in R$ the curvature of A_g can be written as

$$K_g(p) = -y(p)^2 f_s(y(p))$$

where $f_s : (-\epsilon_1, \epsilon_1) \longrightarrow R$ is a C^∞ positive function with $f_s(t) \leq C$ for every $t \in (-\epsilon_1, \epsilon_1)$ (recall the curvature of M is nonpositive). Since $t = t(p)$ is the distance from p to $F(s, 0) = \beta(s)$ we have that

$$K(s,t) \geq -t^2 C$$

where $K(s,t)$ is the curvature of A_g. Let (M_c, g_c) be the family of annulus of step 1. The claim is that there exist $c = c(C) > 0$ and $\delta \leq \epsilon_1$ such that

$$K(s,t) \geq K_{gc}(s,t)$$

for every $(s,t) \in R \times (-\delta, \delta)$. Indeed, if $a > 0$ let $c = \frac{C+a}{12}$. Then we get

$$\begin{aligned} K_{gc}(s,t) &= -(C+a)t^2 \frac{1}{1 + \frac{C+a}{12}t^4} \\ &= -(C+a)t^2\left(1 - \frac{C+a}{12}t^4 + O(t^8)\right) \leq -Ct^2 \end{aligned}$$

for every $(s,t) \in R \times (-\delta_a, \delta_a)$, where $\delta_a > 0$ depends on a. So we conclude that

$$K(s,t) \geq -t^2 C \geq K_{gc}(s,t)$$

for every $(s,t) \in R \times (-\delta_a, \delta_a)$, thus proving the claim for $c = C + a$. Chose any $a > 0$, let $c = C + a$ and denote by A_{gc} the strip $R \times (-\delta, \delta)$ endowed by the metric g_c.

Step 3: Comparison Theory.
A classical reference for this subject is [**4**]. Using the ideas of the CAT-comparison theory we get the following:

Lemma 4.2. *Let $c > 0$, $\delta = \delta_a > 0$, A_g, A_{gc} be as in step 2. Let $\beta(s) = (s, 0)$ be the horizontal line (which is geodesic for both A_g and A_{gc}). Let $p, q \in R \times (-\delta, \delta)$, let $[p, q]_g$, $[p, q]$ be the geodesic segments joining p to q in the metrics g and g_c respectively. Let $\gamma_{s_0}(t) = (s_0, t)$. Then*

$$d_g([p,q]_g \cap \gamma_{s_0}, \beta(s_0)) \geq d_{gc}([p,q] \cap \gamma_{s_0}, \beta(s_0))$$

where d_g is the distance with respect to the metric g.

Lemma 4.2 follows from a standard comparison argument: the curvature K_{gc} is more negative than the curvature K_g in the strip, so distances between geodesics in A_{gc} are more convex than distances between geodesics in A_g. An immediate consequence is the following:

Corollary 4.1. *Let A_g, A_{gc}, $\beta(s)$ be as in lemma 4.2. Let $p \in A_g$ and let $\beta_{g,p}$, $\beta_{gc,p}$ be asymptotic to β starting at p. Then the areas of the ideal triangles*

$$\Delta_g = [\beta(0), p]_g \cup \beta_{g,p}[0, +\infty) \cup \beta[0, +\infty)$$

$$\Delta_{gc} = [\beta(0), p] \cup \beta_{gc,p}[0, +\infty) \cup \beta[0, +\infty)$$

satisfy $area_g(\Delta_g) \geq area_{gc}(\Delta_{gc})$.

Simply because ideal triangles having noncompact parts of β in one of their edges are limits of ordinary triangles having an edge contained in β, so from the previous lemma we deduce that $\Delta_{gc} \subset \Delta_g$. This means that $area_g(\Delta_{gc}) \leq area_g(\Delta_g)$. But from Rauch comparison theorems it is not hard to deduce that

$$area_{gc}(\Delta_{gc}) \leq area_g(\Delta_{gc}) \leq area_g(\Delta_g).$$

Finally, the proof of Lemma 4.1 follows from Theorem 1 and Corollary 4.1.

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